JOURNAL OF PROPULSION AND POWER
Vol. 14, No. 4, July–August 1998

Reflections on Free-Piston Stirling Engines, Part 1: Cyclic Steady Operation

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In spite of the conceptual simplicity, the design of the free-piston Stirling engines (FPSEs) is made difficult by the necessity to accurately foresee the effect of the various geometric, dynamic, and thermodynamic variables on their behavior. This paper describes a fully developed mathematical model able to characterize these machines taking into account all of the relevant physical aspects involved, including the casing motion. A closed-form solution of the governing equations is used, together with a theorem developed by the authors to derive a basic criterion for the FPSE cyclic steady operation. Because this criterion is expressed in an analytical form as a function of the different variables involved, it is possible to choose the engine design parameters to ensure a steady periodic state. The developed model may be used not only for design purposes but also to simulate theoretically the dynamic behavior of a built engine.

Nomenclature

A	= cross-sectional area for moving element
D	= damping coefficient
$\mathcal{D}$	= damping matrix
F	= force
I	= unit matrix
j	= operator defined as $\sqrt{-1}$
L	= Laplace transform operator
L^{-1}	= inverse Laplace transform operator
M	= moving element mass
p	= pressure
R	= residue
Re[s]	= real part of s
r	= displacer-piston stroke ratio, X_d/X_p
r_c	= casing-piston stroke ratio, X_c/X_p
S	= stiffness coefficient
$\mathcal{S}$	= stiffness matrix
S_c	= stiffness of the mounting springs
s	= complex number
s_k	= kth root (eigenvalue) of $W(s)$
t	= time
V	= volume
$\dot{V}$	= volumetric flow rate
$W(s)$	= Wronskian
X	= moving element stroke
x	= displacement
$\mathbf{x}$	= displacement vector
x^*	= displacement defined in the text
α, β, γ, δ	= coefficients defined in the text
Δp	= pressure drop
Δx	= constant defined in the text
μ_{dc}	= displacer-casing mass ratio, M_d/M_c
μ_{pc}	= piston-casing mass ratio, M_p/M_c
ϕ	= piston-displacer phase angle
ϕ_c	= casing-displacer phase angle
Ψ	= pressure drop per unit of volumetric flow rate
ω	= angular frequency

Subscripts

b	= bounce space
c	= compression space, casing
d	= displacer
e	= expansion space
gs	= gas spring
H	= gas spring hysteresis losses
ld–l	= load device–load subsystem
lm	= laminar flow in the regenerator
p	= piston
r	= displacer rod
s	= stop (no-running machine)
tr	= turbulent flow in the heater and cooler
w	= working gas, working gas circuit
0	= initial

Superscripts

$\cdot$	= first-order derivative with respect to time
$\cdot\cdot$	= second-order derivative with respect to time
$-$	= average over a cycle
$'$	= per unit of moving element mass
$\sim$	= Laplace transformed

Introduction

COMPARED with kinematic Stirling engines, the free-piston Stirling engines (FPSEs) of recent design present some inherent features such as 1) a simpler mechanical design because of the absence of the drive mechanism; 2) the possibility to avoid any side force on the piston and, also, by using flexure bearings, the possibility to eliminate all rubbing parts, reducing engine wear[1]; 3) a high-energy conversion (thermal to mechanical) efficiency; and 4) an easy starting and efficient, reliable, and fail-safe power control[2] that make them particularly suitable for those applications for which the highest reliability is basic.

These applications include, for instance, remote area power generators, space power systems, and dish solar terrestrial sys-

Received July 31, 1997; revision received Feb. 7, 1998; accepted for publication Feb. 16, 1998.

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tems, and they are characterized by requirements of high performance, long life, and maintenance-free operation.

The most common arrangement used for these applications is a single-cylinder FPSE connected to a linear alternator (LA) for the production of electricity from fuel and solar and nuclear energy.

Engines powered with gaseous or liquid fuel, as well as with biomass fuel, have been proposed for remote power generators or domestic cogeneration plants.[2] For space power systems the use of solar[3] and nuclear[4–7] energy has been considered, depending on the mission goal.

Solar electric generation systems for terrestrial applications use a parabolic dish concentrator with an FPSE/LA at the focus. They are currently being developed and are approaching commercialization.[8] These systems combine the efficiency of the dynamic systems with the reliability of the static energy-conversion systems, such as thermoelectric and photovoltaic modes.

However, optimization of the dynamic behavior of the free-piston machine is not a simple matter, because of the lack of mechanical linkages able to fix strokes (X_p, X_d) and phasing (ϕ) for the moving elements. To this aim, it is required to carefully analyze 1) the cyclic steady state of the machine and, subsequently, 2) its stable operation when the power controls (either internal or external) are inoperable. This is a very important step in the initial stages of the design because the FPSE configuration often has a significant bearing on this question. The former analysis concerns the first part of this study while the latter is considered in the second part.[9]

Because the laws of motion of the moving elements of free-piston Stirling machines are not known presumptively, as in case of kinematic Stirling machines, they have to be calculated by means of an accurate dynamic analysis. If the equations of motion are linearized (linear dynamic analysis), the result is a system of linear and homogeneous differential equations with constant coefficients. These equations may be solved analytically either in the time domain or in the Laplace domain. Solution of the polynomial characteristic equation yields the conditions for the periodic steady operation of the FPSE/load device–load (LD–L) system. In addition, very useful algebraic expressions linking the various machine parameters may be derived. In fact, they may be used for both the performance prediction and the design of the FPSE/LD–L system, as will be shown in the second part of this study.[9]

Dynamic Balance Equations

The equations that describe the dynamic behavior of an FPSE connected to a generic LD–L subsystem can be written as follows (Fig. 1):

$$M_p\ddot{x}_p = (p_c - p_b)A_p - D_{b,H}(\dot{x}_p - \dot{x}_c) + F_{\text{ld-l}} \tag{1}$$

$$M_d\ddot{x}_d = (p_c - p_{\text{gs}})A_r + (p_e - p_c)A_d - D_{\text{gs},H}(\dot{x}_d - \dot{x}_c) \tag{2}$$

$$M_c\ddot{x}_c = (p_b - p_c)A_p + (p_{\text{gs}} - p_c)A_r + (p_c - p_e)A_d - D_{b,H}(\dot{x}_c - \dot{x}_p) - D_{\text{gs},H}(\dot{x}_c - \dot{x}_d) - F_{\text{ld-l}} - S_c x_c \tag{3}$$

Therefore, the listed equations of motion are valid: 1) For a single-cylinder (beta) engine with displacer sprung to ground, where the displacer is completely resonated by a gas spring[10,11]; and 2) for an engine adopting a clearance seal/centering port system for the power piston and displacer.[2,10,11]

The pressures p_e and p_c, which appear in the equations of motion [Eqs. (1–3)], are given in the Pressures in Expansion and Compression Spaces section.

p_{gs} and p_b are ideal pressures because they are not affected by the hysteresis losses that take place in the gas spring and bounce space of the engine. The effect of these losses is taken into account by means of the $D_{\text{gs},H}$ and $D_{b,H}$ damping coefficients, which may be calculated as indicated in Ref. 12 if the machine is running at a cyclic steady state.

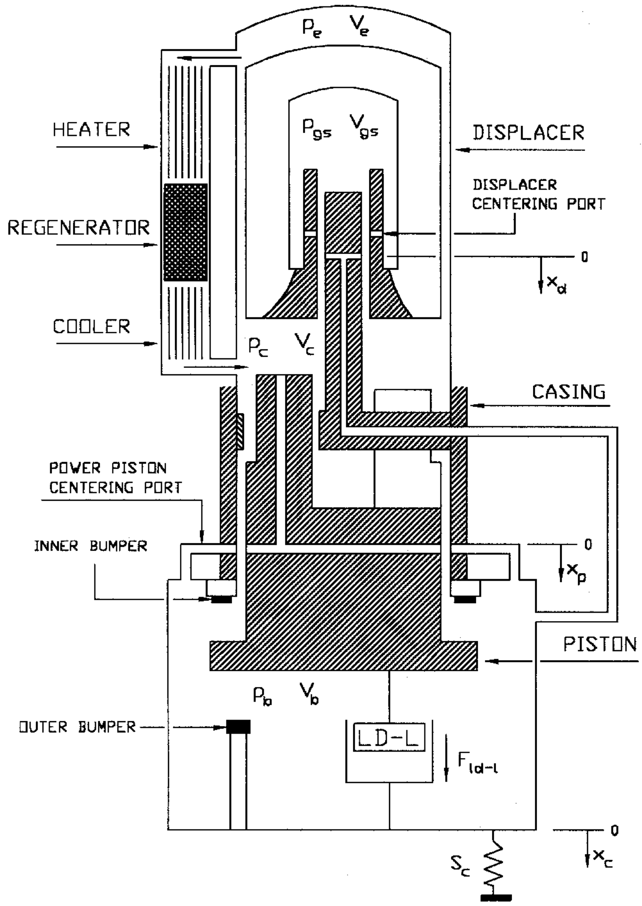

Fig. 1 FPSE.

If the displacer uses planar mechanical springs in place of pressure-dependent gas springs,[2] the stiffness force exerted on the displacer motion by the metallic springs has to be considered in Eqs. (2) and (3) in place of the pressure force $p_{\text{gs}}A_r$ and the damping force $\pm D_{\text{gs},H}(\dot{x}_d - \dot{x}_c)$.

Also, ring seals will not be considered because present practice indicates a preference for clearance seals.[1,2] However, in those cases where ring seals are still used, as in Martini-type FPSEs,[13] the seal leakage is usually very small, but in this type of situation the mechanical friction is generally of much greater importance. Thus, the mechanical friction forces on the power piston, displacer, and casing must be considered in Eqs. (1–3), as shown in Ref. 13.

The behavior of either external or internal power controls has not been included in the dynamic model, because we are interested in the natural response of the machine to load changes and to other variations in the operating parameters. This is a very important step in the initial stages of the design because it allows the self-regulating capability of an FPSE, that is, the stabilizing effect of the various nonlinear thermo-fluid-dynamic terms inherent to the machine's operation, to be analyzed.

The effects of the gravitational field have also been neglected because they are quite small compared to the effects of the pressures involved in the machine. Obviously, the gravitational field may be fully ignored in case of space power applications.

The term $F_{\text{ld-l}}$, which appears in Eqs. (1) and (3), represents the force exerted on the motions of both the power piston and casing by the LD–L subsystem connected to the engine (LA, hydraulic pump, compressor). In electric generation systems, the LD–L subsystem is represented by an LA connected through a series tuning capacitor to a static load, which for a machine that is running at a cyclic steady-state produces a force depending linearly on the power piston displacement x_p and velocity $\dot{x}_p$, and on the casing displacement x_c and velocity $\dot{x}_c$.[14,15] However, in FPSE/LA systems the LA may also be connected to a dynamic load.[16]

In writing the equations of motion [Eqs. (1–3)], it has been assumed that for each oscillating part (power piston, displacer, and casing) the origin of the space coordinates x_p, x_d, and x_c is located at the midpoint of the stroke. In fact, the centers of the piston, displacer, and casing strokes do not change during the engine operation if the machine uses a clearance seal/centering device system for the moving elements,[10,11] and an ideal performance of the centering devices is assumed in the dynamic analysis. Therefore, the adopted frame of reference may be considered to be inertial.

If the engine uses a clearance seal/flexure bearing system for piston and displacer[1] in place of the clearance seal/centering port system, the adopted frame of reference is not inertial and suitable inertia forces should be introduced in Eqs. (1–3), unless a different frame of reference is adopted. In fact, the clearance seal/flexure bearing system almost eliminates the seal leakage losses around the power piston and displacer rod, holding the midpoints of the strokes of the moving elements at a fixed location when the machine is running at a cyclic steady state. Any variation in the operating conditions, i.e., the increase of the engine hot-end temperature during its starting, modifies the average pressure of the working gas circuit, displacing the mean positions of the piston, displacer, and casing toward new locations.

Therefore, for a machine using a clearance seal/flexure bearing system, the frame of reference characterized by the origin of the space coordinates located at the midpoint of the stroke may be considered inertial only during its periodic steady operation. In this case, the equations of motion [Eqs. (1–3)], written for a machine using a clearance seal/centering port system, are still valid provided the space coordinates, x_p, x_d, and x_c are replaced by the following:

$$x_p^* = x_p + \Delta x_p, \qquad x_d^* = x_d + \Delta x_d, \qquad x_c^* = x_c + \Delta x_c \tag{4}$$

where Δx_p, Δx_d, and Δx_c represent the locations of the midpoints of the moving elements strokes at a generic cyclic steady state with respect to their locations when the machine is not running.

The positions Δx_p, Δx_d, and Δx_c are time independent and may be evaluated taking the following facts into account:

1) The mean pressures within the different engine spaces (working gas circuit, gas spring, bounce space) are equal in value, i.e.,

$$\bar{p}_w = \bar{p}_{gs}, \qquad \bar{p}_w = \bar{p}_b$$

2) The sum of the average gas volumes within the different engine spaces (expansion and compression spaces, gas spring, bounce space) is constant when changing operating conditions and is equal to the sum of the gas volumes when the machine is at rest, i.e.,

$$\bar{V}_e + \bar{V}_c + \bar{V}_{gs} + \bar{V}_b = V_{e,s} + V_{c,s} + V_{gs,s} + V_{b,s}$$

The average volumes may be evaluated as shown in the following text.

Pressures in Expansion and Compression Spaces

The pressure difference between the expansion and compression spaces, $p_e - p_c$, appearing in Eqs. (2) and (3), is a result of the viscous flow losses and so-called "minor losses" (sudden expansion and contraction, bends, etc.) through the heat exchangers and regenerator of the working gas circuit of the machine. These pressure losses, indicated with the symbol Δp_w, are in many cases the dominant engine losses.[10,17] By considering the gas flow from the expansion space to the compression one as positive (Fig. 1), we have

$$p_e - p_e = \Delta p_w > 0$$

The p_e and p_c pressures that appear in Eqs. (1–3) are calculated assuming the following position:

$$p_e = p_w + \Delta p_w/2, \qquad p_e = p_w - \Delta p_w/2 \tag{5}$$

where p_w is the cycle gas pressure evaluated neglecting the pressure drop Δp_w. Thus, p_w is an ideal pressure, because it is not affected by the energy losses caused by Δp_w.

Because the pressure drop in the working gas circuit of the machine is taken into consideration, it is physically wrong to define a pressure, p_w, for the entire working gas circuit. However, in this dynamic analysis the pressure drop has to be considered as an element resisting the piston and displacer motions, and not as an element that tends to remove the spatial uniformity of the pressure in the cycle fluid circuit.

In Stirling machines the working gas flow through the heat exchangers and regenerator is unsteady, because it oscillates between the expansion and compression spaces. Thus, assuming the flow as one-dimensional, Δp_w can be expressed in the following form:

$$\Delta p_w = \Psi_{lm}\dot{V}_w + \Psi_{tr}\dot{V}_w|\dot{V}_w| \tag{6}$$

where the linear term, $\Psi_{lm} \cdot \dot{V}_w$, is because of the laminar viscous flow losses that mainly take place within the regenerator as a result of its quite small hydraulic radius. Instead, the nonlinear term of second-order, $\Psi_{tr} \cdot \dot{V}_w \cdot |\dot{V}_w|$, is because of the turbulent viscous flow and minor losses, which mainly take place within the heat exchangers and adjacent volumes' connecting ducts.

The quantities Ψ_{lm} and Ψ_{tr} (time independent) may be calculated once the geometry of the heat exchangers and regenerator is known as well as the thermal properties of the working gas. The term $\dot{V}_w$ is the volumetric flow rate through the heat exchangers and regenerator and may be calculated as a direct consequence of the position [Eq. (5)] assumed for p_e and p_c, according to energy balance considerations.

The starting point in the evaluation of $\dot{V}_w$ is that the energy lost over a cycle because of Δp_w can be calculated in two different ways, to obviously get the same result.

Starting from the equations of motion [Eqs. (1–3)], with the position of Eq. (5), we get the following expression (written in positive terms) for the cyclic energy lost because of Δp_w:

$$\frac{1}{2}\oint \Delta p_w[A_p(\dot{x}_p - \dot{x}_c) - (2A_d - A_r)(\dot{x}_d - \dot{x}_c)]\,\mathrm{d}t \tag{7}$$

Alternately, having the gas flow through the heat exchangers and regenerator $\dot{V}_w$ as a volumetric flow rate, the energy dissipated over a cycle (written in positive terms) is given by (work = force × displacement):

$$\oint \Delta p_w \dot{V}_w\,\mathrm{d}t \tag{8}$$

Comparing Eqs. (7) and (8) yields the following expression for the volumetric flow rate:

$$\dot{V}_w = \tfrac{1}{2}[A_p(\dot{x}_p - \dot{x}_c) - (2A_d - A_r)(\dot{x}_d - \dot{x}_c)] \tag{9}$$

Referring to Fig. 1, the volumes of the expansion and compression spaces are

$$V_e = V_{e,s} + A_d(x_d - x_c) \tag{10}$$

$$V_c = V_{c,s} + A_p(x_p - x_c) - (A_d - A_r)(x_d - x_c) \tag{11}$$

Equations (10) and (11) allow Eq. (9) to be written as a function of the time-variations of the expansion and compression volumes:

$$\dot{V}_w = \frac{1}{2}\left(\frac{dV_c}{dt} - \frac{dV_e}{dt}\right) \tag{12}$$

Therefore, the position of Eq. (5) for p_e and p_c is physically consistent only with Eq. (12) for $\dot{V}_w$. Thus, the effect of Δp_w on the laws of motion of the reciprocating elements is adequately taken into account. Equation (12) shows that the gas flow moves from the expansion space to the compression one, according to the authors' assumption.

Volumes of the Time-Dependent Spaces

Referring to Fig. 1, the volumes of the expansion and compression spaces are given by Eqs. (10) and (11), respectively, whereas the volumes of the gas spring and bounce space are

$$V_{gs} = V_{gs,s} - A_r(x_d - x_c) \tag{13}$$

$$V_b = V_{b,s} - A_p(x_p - x_c) \tag{14}$$

It may be noted that Eqs. (10) and (11) for V_e and V_c are valid for any beta free-piston machine, including the thermomechanical generator (TMG), where $A_r = 0$,[10,15] and not only for those machines with the displacer sprung to ground, to which we refer in this study. In fact, the cycle fluid circuit does not depend on the beta engine arrangement: 1) common gas spring for piston and displacer, 2) displacer sprung to ground, and 3) displacer sprung to piston. On the contrary, Eqs. (13) and (14) for V_{gs} and V_b are valid only for FPSEs with displacer sprung to ground.

When the machine is running at a cyclic steady state, which depends on the adopted operating conditions, the laws of motion of the moving elements are periodic and steady functions. If a clearance seal/centering device system is used, the centers of oscillation of the piston, displacer, and casing do not change during the machine working, and the laws of motion are functions with zero mean value. Therefore, the following relations are valid for the cyclic average values of the volumes:

$$\bar{V}_e = V_{e,s}, \qquad \bar{V}_c = V_{c,s}, \qquad \bar{V}_{gs} = V_{gs,s}, \qquad \bar{V}_b = V_{b,s}$$

If a clearance seal/flexure bearing system is used in the machine, Eqs. (10), (11), (13), and (14) have to be rewritten bearing in mind the position [Eq. (4)]. In this case, the laws of motion of the moving elements, x_p, x_d, and x_c are still periodic and steady functions with zero mean value during the cyclic operation and, therefore, the following relations are valid:

$$\bar{V}_e = V_{e,s} + A_d(\Delta x_d - \Delta x_c)$$

$$\bar{V}_c = V_{c,s} + A_p(\Delta x_p - \Delta x_c) - (A_d - A_r)(\Delta x_d - \Delta x_c)$$

$$\bar{V}_{gs} = V_{gs,s} - A_r(\Delta x_d - \Delta x_c), \qquad \bar{V}_b = V_{b,s} - A_p(\Delta x_p - \Delta x_c)$$

Casing Motion

The casing motion $x_c(t)$ has been considered within the dynamic equation set developed in the preceding sections.

In fact, while it is true that in many cases the casing motion can be neglected because its amplitude is considerably smaller than both the piston and displacer amplitudes,[2,8,18] it may be inadvisable to ignore these motions in all cases. For example, the advent of high-performance, space-based FPSE systems,[4–7] and in some cases ground-based units such as TMGs,[10,15] has placed a premium on high efficiency and minimum system mass along with minimal vibration-induced loads transmitted to the spacecraft or other supporting structure.

Under these conditions it becomes critical that the casing amplitudes be incorporated into the dynamic equations as a result of the strong feedback effect of casing motion on the displacer-operating characteristics. This in turn will effect power output of the FPSE, can influence system stability, and will play a role in the vibration-induced forces transmitted to the spacecraft. Therefore, the FPSE system has to be considered, from a dynamic point of view, as a system having three degrees of freedom, i.e., piston, displacer, and casing.

However, Eqs. (1–3) may be reduced to only two equations because it is possible to find, by means of a reasonable approximation, a relation linking the laws of motion of the piston, displacer, and casing; namely, $x_p(t)$, $x_d(t)$, and $x_c(t)$. To this end, let us consider the equation of motion of the machine's c.m. that can be obtained also by summing both sides of Eqs. (1), (2), and (3):

$$M_p\ddot{x}_p + M_d\ddot{x}_d + M_c\ddot{x}_c + S_cx_c = 0 \tag{15}$$

The force exerted by S_c is small compared to the inertia force, so that the whole engine can be treated as a free-body.[15] Thus, Eq. (15) may be written in the following form:

$$\frac{d}{dt}(M_p\dot{x}_p + M_d\dot{x}_d + M_c\dot{x}_c) = 0$$

showing that the entire engine momentum is constant and is zero, because the moving elements are initially at rest. Thus, the following relation is obtained:

$$x_c = -\mu_{pc}x_p - \mu_{dc}x_d \tag{16}$$

Substituting Eq. (16) into Eqs. (1) and (2), we get a system of only two differential equations:

$$M_p\ddot{x}_p = (p_c - p_b)A_p - D_{b,H}[(1 + \mu_{pc})\dot{x}_p + \mu_{dc}\dot{x}_d] + F_{ld-l} \tag{17}$$

$$M_d\ddot{x}_d = (p_c - p_{gs})A_r - D_{gs,H}[(1 + \mu_{dc})\dot{x}_d + \mu_{pc}\dot{x}_p] + (p_e - p_c)A_d \tag{18}$$

By solving Eqs. (17) and (18), the piston and displacer laws of motion may be obtained. From those, Eq. (16) allows the casing law of motion to be calculated. Having three degrees of freedom, the FPSE/LD–L system can be analytically still described as a dynamic system with only two degrees of freedom. Equations (9–11), (13), and (14), may be rewritten taking into account Eq. (16).

Operational Parameters

The operational parameters are those parameters that may be varied separately from others to control the engine power. For an FPSE/LD–L system they are 1) inlet temperature of the heating fluid, 2) volumetric flow rate of the heating fluid, 3) inlet temperature of the cooling fluid, 4) volumetric flow rate of the cooling fluid, 5) load, and 6) total mass of the working fluid (working gas circuit + gas spring + bounce space).

If the listed operational parameters do not change during the engine working, i.e., they are time independent, then the oscillations of the moving elements are stationary provided the engine has been properly designed. In other words, the time independence of the operating conditions does not ensure that the machine is a steady oscillator. Therefore, this time independence may be only considered as a necessary condition for the periodic steady operation. The fundamental condition for this operation remains to be found, and its study represents the main purpose of this work. In addition, if the oscillations are steady the operating conditions must assume suitable values so that the piston and displacer amplitudes are consistent with the geometric constraints of the machine.

If one or more operational parameters of a properly designed FPSE change during the engine running, i.e., they are time dependent, a transient working occurs. In this case we can have three different situations: 1) starting, characterized by divergent oscillations of the piston and displacer; 2) stopping, characterized by convergent oscillations; and 3) moving toward a new cyclic steady state because of a change of at least one of the machine operating parameters, generally to a load change (among the listed operating conditions the load is usually of greater concern). In the latter situation, the oscillations of the moving elements can be either convergent or divergent until a new periodic steady state is reached.

The equations of motion [Eqs. (17) and (18)], with position (5), have been written in a general way, that is they are valid for both time-independent and time-dependent operational parameters. In the former case, the variables appearing in the dynamic equations, p_w, p_{gs}, p_b, Δp_w, and F_{ld-l}, do not depend on the time t explicitly, but only implicitly by means of x_p, $\dot{x}_p$ and x_d, $\dot{x}_d$. In the latter case, the variables listed before depend on t both explicitly and implicitly.

If the engine uses a clearance seal/centering device system, the study of the transient working is quite difficult. In this case the gas masses within the working gas circuit, the bounce space, and the gas spring are time dependent during the transient response[14] (Appendix A).

If the engine, instead, uses a clearance seal/flexure bearing system, the study of the transient is still quite difficult because the adopted frame of reference is not inertial, and suitable inertia forces should be considered in the engine dynamic equations.

In addition, the expressions of the $D_{gs,H}$ and $D_{b,H}$ damping coefficients given in Ref. 12, as well as the expression of the linear alternator force given in Refs. 14 and 15, are valid only for the cyclic steady operation of the machine.

Therefore, only the operation of the machine with constant operating conditions will be considered here.

Linearized Dynamic Balance Equations

Because the functional dependences on x_p, $\dot{x}_p$, and x_d, $\dot{x}_d$ assume a nonlinear form for the variables p_w, p_{gs}, p_b, Δp_w, and F_{ld-l} appearing in the dynamic equations, Eqs. (17) and (18), with position (5), are nonlinear differential equations whose solution defines the behavior of the FPSE/LD–L system. The functional dependence on x_c and $\dot{x}_c$ does not appear because of relation (16).

Applying a suitable linearization technique, described in a companion paper,[9] the functions p_w, p_{gs}, p_b, Δp_w, and F_{ld-l} may be replaced by their linearized expressions obtaining the following dynamic equations:

$$\ddot{x}_p + D'_{pp}\dot{x}_p + D'_{pd}\dot{x}_d + S'_{pp}x_p + S'_{pd}x_d = 0 \qquad (19)$$

$$\ddot{x}_d + D'_{dp}\dot{x}_p + D'_{dd}\dot{x}_d + S'_{dp}x_p + S'_{dd}x_d = 0 \qquad (20)$$

where the right-hand terms are equal to zero because the average pressures $\bar{p}_w$, $\bar{p}_{gs}$, and $\bar{p}_b$ are equal in value.

The adoption of a linearization technique for the equations of motion of the moving elements allows one to analytically solve these equations and, thus, to find useful algebraic relations. These relations may be used not only for the performance prediction, but also for the design of the engine in order to ensure a cyclic steady and stable operation, as well as a good general performance.

Equations (19) and (20) are valid for any beta engine, whereas the S'_{ij} and D'_{ij} coefficients ($i = p, d$; $j = p, d$), defined in Appendix B, depend on the beta engine arrangement. This is because volumes V_b and V_{gs} and, therefore, the pressures p_b and p_{gs}, depend on x_p and x_d in a different way according to the considered beta engine arrangement.

The S'_{ij} and D'_{ij} coefficients represent the stiffness and damping coefficients per unit of moving element mass of the FPSE/LD–L system. For example, coefficient S'_{pd} accounts for the influence on the piston motions by virtue of a spring coupling to the motions of the displacer, whereas S'_{pp} is a spring effect unique to the piston. Similarly, coefficient D'_{dp} accounts for the influence on the displacer motions by virtue of a dashpot coupling to the motions of the piston, whereas D'_{dd} is a dashpot effect unique to the displacer.

Because the developed analysis considers the operational parameters of the machine independent of the time, the S'_{ij} and D'_{ij} coefficients are constant and, as shown in a companion paper, they account for 1) the thermodynamic behavior of the engine and 2) the influence of the adopted load device on the engine operation.

Equations (19) and (20) may also be written in matrix form:

$$\begin{bmatrix}\ddot{x}_p\\ \ddot{x}_d\end{bmatrix} + \begin{bmatrix}D'_{pp} & D'_{pd}\\ D'_{dp} & D'_{dd}\end{bmatrix}\cdot\begin{bmatrix}\dot{x}_p\\ \dot{x}_d\end{bmatrix} + \begin{bmatrix}S'_{pp} & S'_{pd}\\ S'_{dp} & S'_{dd}\end{bmatrix}\cdot\begin{bmatrix}x_p\\ x_d\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$

In a compact manner, we have

$$\ddot{\boldsymbol{x}} + \mathcal{D}'\dot{\boldsymbol{x}} + \mathcal{S}'\boldsymbol{x} = 0 \qquad (21)$$

where $\mathcal{D}'$ and $\mathcal{S}'$ are, respectively, the damping and stiffness matrices per unit of moving element mass of the machine, whereas $\boldsymbol{x}(t)$, $\dot{\boldsymbol{x}}(t)$, and $\ddot{\boldsymbol{x}}(t)$ are, respectively, the displacement, velocity, and acceleration vectors.

The differential model [Eq. (21)] formally represents a damped dynamic system (periodic or aperiodic) having two degrees of freedom. It represents a self-excitating dynamic system, whose self-excitation is a result of the asymmetry of the $\mathcal{S}'$ matrix.[18]

Closed-Form Solution of the Linearized Dynamic Balance Equations

Because Eq. (21) represents a system of two homogeneous and linear second-order differential equations with constant coefficients, it can be solved analytically either in the time domain or in the Laplace domain.

Integration in the Time Domain

The general solution of the differential matrix equation [Eq. (21)] is given by a linear combination of particular solutions having the following form[19]:

$$\boldsymbol{q}_k = \boldsymbol{f}_k e^{s_k t} = \begin{bmatrix}f_{1k}\\ f_{2k}\end{bmatrix} e^{s_k t} \qquad (22)$$

Thus, we have

$$\boldsymbol{x}(t) = \sum_{k=1}^{n} c_k \boldsymbol{q}_k = \sum_{k=1}^{n} c_k \boldsymbol{f}_k e^{s_k t} \qquad (23)$$

where s_k is the kth root (eigenvalue) of the nth degree characteristic polynomial associated with the system [Eq. (21)], called Wronskian and indicated with the symbol $W(s)$:

$$W(s) = \det(Is^2 + \mathcal{D}'s + \mathcal{S}') \qquad (24)$$

The vector $\boldsymbol{f}_k$ is the kth eigenvector associated with the eigenvalue s_k, whereas the constant c_k is the kth coefficient of the linear combination [Eq. (23)]. Note that the solution of the matrix differential equation [Eq. (21)] has the form of Eq. (23) only in cases of eigenvalues with a multiplicity equal to one.

Bearing in mind the expressions of $\mathcal{D}'$ and $\mathcal{S}'$ matrices, which are 2 by 2 matrices, the Wronskian defined by Eq. (24) may be written as follows:

$$W(s) = Y_p(s)Y_d(s) - \lambda_p(s)\lambda_d(s) \tag{25}$$

where

$$Y_p(s) = s^2 + D'_{pp}s + S'_{pp} \tag{26}$$

$$Y_d(s) = s^2 + D'_{dd}s + S'_{dd} \tag{27}$$

$$\lambda_p(s) = D'_{pd}s + S'_{pd} \tag{28}$$

$$\lambda_d(s) = D'_{dp}s + S'_{dp} \tag{29}$$

The associated characteristic polynomial is fourth-degree with real coefficients ($\Rightarrow n = 4$). Bearing in mind that s_k is the kth root of this polynomial, and $\boldsymbol{f}_k$ is the eigenvector associated with it, the calculation of $\boldsymbol{f}_k$ may be obtained by requiring that the vectorial function [Eq. (22)] is the solution of the matrix differential equation [Eq. (21)]

$$(Is_k^2 + \mathcal{D}'s_k + \mathcal{S}')\boldsymbol{f}_k e^{s_k t} = 0$$

Because $e^{s_k t} > 0 \ \forall \ t$, we have the following matrix algebraic equation:

$$(Is_k^2 + \mathcal{D}'s_k + \mathcal{S}')\boldsymbol{f}_k = 0$$

which may also be written as follows:

$$Y_p(s_k)f_{1k} + \lambda_p(s_k)f_{2k} = 0$$

$$\lambda_d(s_k)f_{1k} + Y_d(s_k)f_{2k} = 0$$

Thus, we have two homogeneous linear algebraic equations in two unknowns, f_{1k} and f_{2k}. We obtain the solution $(f_{1k}, f_{2k}) \neq (0, 0)$ only when

$$Y_p(s_k)Y_d(s_k) - \lambda_p(s_k)\lambda_d(s_k) = 0$$

This equation is certainly satisfied because s_k is the kth root of the characteristic polynomial [Eq. (25)]. Therefore, the eigenvector $\boldsymbol{f}_k$ associated with the eigenvalue s_k is

$$\boldsymbol{f}_k = \begin{bmatrix} f_{1k} \\ f_{2k} \end{bmatrix} = \begin{bmatrix} -\dfrac{\lambda_p(s_k)}{Y_p(s_k)} \\ 1 \end{bmatrix} = \begin{bmatrix} -\dfrac{Y_d(s_k)}{\lambda_d(s_k)} \\ 1 \end{bmatrix}$$

Bearing in mind Eq. (23), with $n = 4$, the laws of motion of the moving elements are

$$x_p(t) = -c_1 \frac{\lambda_p(s_1)}{Y_p(s_1)} e^{s_1 t} - c_2 \frac{\lambda_p(s_2)}{Y_p(s_2)} e^{s_2 t} - c_3 \frac{\lambda_p(s_3)}{Y_p(s_3)} e^{s_3 t} - c_4 \frac{\lambda_p(s_4)}{Y_p(s_4)} e^{s_4 t} \tag{30}$$

$$x_d(t) = c_1 e^{s_1 t} + c_2 e^{s_2 t} + c_3 e^{s_3 t} + c_4 e^{s_4 t} \tag{31}$$

where the coefficients c_1, c_2, c_3, and c_4 may be evaluated by imposing the initial conditions for the piston and displacer:

$$x_p(t = 0) = x_{p0}, \qquad x_d(t = 0) = x_{d0}$$

$$\dot{x}_p(t = 0)\ \dot{x}_{p0}, \qquad \dot{x}_d(t = 0) = \dot{x}_{d0}$$

Integration in the Laplace Domain

The differential matrix equation [Eq. (21)] may also be solved in the Laplace domain, obtaining the same results. If the Laplace transform applies to the $\boldsymbol{x}(t)$, $\dot{\boldsymbol{x}}(t)$, and $\ddot{\boldsymbol{x}}(t)$ vectors, we have

$$L[\boldsymbol{x}(t)] = \tilde{\boldsymbol{x}}(s)$$

$$L[\dot{\boldsymbol{x}}(t)] = s\tilde{\boldsymbol{x}}(s) - \boldsymbol{x}(t = 0) = s\tilde{\boldsymbol{x}}(s) - \boldsymbol{x}_0$$

$$L[\ddot{\boldsymbol{x}}(t)] = s^2\tilde{\boldsymbol{x}}(s) - s\boldsymbol{x}(t = 0) - \dot{\boldsymbol{x}}(t = 0) = s^2\tilde{\boldsymbol{x}}(s) - s\boldsymbol{x}_0 - \dot{\boldsymbol{x}}_0$$

where s is the Laplace transform variable. Thus, taking the Laplace transform of both sides of Eq. (21) and rearranging it to make $\tilde{\boldsymbol{x}}(s)$ the subject, we obtain

$$\tilde{\boldsymbol{x}}(s) = (Is^2 + \mathcal{D}'s + \mathcal{S}')^{-1}[(\mathcal{D}' + Is)\boldsymbol{x}_0 + \dot{\boldsymbol{x}}_0]$$

Therefore,

$$\begin{bmatrix} \tilde{x}_p(s) \\ \tilde{x}_d s) \end{bmatrix} = \begin{bmatrix} Y_p(s) & \lambda(s) \\ \lambda_d(s) & Y_d(s) \end{bmatrix}^{-1} \cdot \begin{bmatrix} \zeta_{p0} \\ \zeta_{d0} \end{bmatrix} \tag{32}$$

where $Y_p(s)$, $Y_d(s)$, $\lambda_p(s)$, and $\lambda_d(s)$ are given, respectively, by Eqs. (26–29), and

$$\zeta_{p0}(s) = (D'_{pp} + s)x_{p0} + D'_{pd}x_{d0} + \dot{x}_{p0} \tag{33}$$

$$\zeta_{d0}(s) = D'_{dp}x_{p0} + (D'_{dd} + s)x_{d0} + \dot{x}_{d0} \tag{34}$$

With suitable manipulations, Eq. (32) allows the laws of motion of the piston and displacer in the Laplace domain to be calculated

$$\tilde{x}_p(s) = \frac{Y_d(s)\zeta_{p0}(s) - \lambda_p(s)\zeta_{d0}(s)}{W(s)} \tag{35}$$

$$\tilde{x}_d(s) = \frac{Y_p(s)\zeta_{d0}(s) - \lambda_d(s)\zeta_{p0}(s)}{W(s)} \tag{36}$$

where $W(s)$ is given by Eq. (25). To obtain the laws of motion in the time domain, the inverse Laplace transform must be applied to the functions $\tilde{x}_p(s)$ and $\tilde{x}_d(s)$:

$$L^{-1}[\tilde{x}_p(s)] = x_p(t), \qquad L^{-1}[\tilde{x}_d(s)] = x_d(t)$$

In this case, however, the functions $\tilde{x}_p(s)$ and $\tilde{x}_d(s)$, whose inverse are required, are not recognizable as standard types, such as those listed in the tables of inverse Laplace transforms.[20] In particular, in the algebraic expressions of $\tilde{x}_p(s)$ and $\tilde{x}_d(s)$, the numerator is of degree 3 while the denominator, $W(s)$, is of degree 4.

Then, by using partial fractions as shown in Appendix A, it is possible to resolve the functions $\tilde{x}_p(s)$ and $\tilde{x}_d(s)$, given by Eqs. (35) and (36), into simpler fractions that may be inverted on sight from tables.

If all of the roots of s_k ($k = 1, \ldots, 4$) of $W(s)$ polynomial have multiplicity $m_k = 1$, Eqs. (35) and (36) may be written [applying Eq. (A1), as shown in Appendix A]

$$\tilde{x}_p(s) = \frac{R_{p1}}{s - s_1} + \frac{R_{p2}}{s - s_2} + \frac{R_{p3}}{s - s_3} + \frac{R_{p4}}{s - s_4} \tag{37}$$

$$\tilde{x}_d(s) = \frac{R_{d1}}{s - s_1} + \frac{R_{d2}}{s - s_2} + \frac{R_{d3}}{s - s_3} + \frac{R_{d4}}{s - s_4} \tag{38}$$

where the residues R_{pk} and R_{dk} associated with the s_k eigenvalue may be calculated by means of Eq. (A2) as indicated in Appendix B. Obviously, they depend on the initial conditions of the piston and displacer appearing in Eqs. (35) and (36).

From the tables of inverse Laplace transform[20] it follows that

$$L^{-1}[1/(s - s_k)] = e^{s_k t}$$

and, therefore, the inverse Laplace transforms of Eqs. (37) and (38) give the laws of motion of the moving elements in the time domain

$$x_p(t) = R_{p1}e^{s_1 t} + R_{p2}e^{s_2 t} + R_{p3}e^{s_3 t} + R_{p4}e^{s_4 t} \tag{39}$$

$$x_d(t) = R_{d1}e^{s_1 t} + R_{d2}e^{s_2 t} + R_{d3}e^{s_3 t} + R_{d4}e^{s_4 t} \tag{40}$$

If one root of $W(s)$ polynomial, for example s_1, has multiplicity $m_1 = 3$, then Eqs. (35) and (36) become [applying Eq. (A3), as shown in Appendix A]

$$\tilde{x}_p(s) = \frac{R_{p11}}{s - s_1} + \frac{R_{p12}}{(s - s_1)^2} + \frac{R_{p13}}{(s - s_1)^3} + \frac{R_{p2}}{s - s_2}$$

$$\tilde{x}_d(s) = \frac{R_{d11}}{s - s_1} + \frac{R_{d12}}{(s - s_1)^2} + \frac{R_{d13}}{(s - s_1)^3} + \frac{R_{d2}}{s - s_2}$$

where the residues $R_{p1\alpha}$ and $R_{d1\alpha}$ associated with the s_1 eigenvalue ($\alpha = 1, \ldots, m_1 = 3$) may be calculated by means of Eq. (A4) as indicated in Appendix A.

From the tables of inverse Laplace transform[20]

$$L^{-1}\left[\frac{1}{(s - s_k)^\alpha}\right] = \frac{t^{\alpha-1}e^{s_k t}}{(\alpha - 1)!}$$

Therefore

$$x_p(t) = R_{p11}e^{s_1 t} + R_{p12}te^{s_1 t} + R_{p13}(t^2/2)e^{s_1 t} + R_{p2}e^{s_2 t}$$

$$x_d(t) = R_{d11}e^{s_1 t} + R_{d12}te^{s_1 t} + R_{d13}(t^2/2)e^{s_1 t} + R_{d2}e^{s_2 t}$$

where

$$\lim_{t\to\infty} te^{s_1 t} = \lim_{t\to\infty}(t^2/2)e^{s_1 t} = \begin{cases} \infty & \text{if} \quad \mathrm{Re}[s_1] > 0 \\ 0 & \text{if} \quad \mathrm{Re}[s_1] < 0 \end{cases}$$

As observed earlier, the integration of Eqs. (19) and (20) in the Laplace domain leads to the same results obtained previously by the integration in the time domain. In other words, it is not essential to apply the Laplace transform to obtain the FPSE/LD–L characteristic polynomial $W(s)$, whose roots affect the engine operation.

Cyclic Steady Operation

As discussed in the previous section, it is clear that the laws of motion of the moving elements, given by Eqs. (30) and (31) as well as by Eqs. (39) and (40), are closely linked to the form of the roots of the $W(s)$ characteristic polynomial associated with the FPSE/LD–L system. Such roots obviously depend on the damping D'_{ij} and stiffness S'_{ij} coefficients of the machine, as shown by Eq. (25), with the positions of Eqs. (26–29). According to the form of these roots s_k ($k = 1, \ldots, 4$), we have the following cases.

1) Roots s_k with a negative real part: $\mathrm{Re}[s_k] < 0$. In this case, the exponential functions that appear in the laws of motion [Eqs. (39) and (40)] of the piston and displacer tend to zero when the time increases. In general, there is not any oscillation. However, if s_1 and s_2 are two complex conjugate roots: $s_1, s_2 = \sigma \pm j\omega$, where σ is negative, then, neglecting the transient relative to the two roots s_3 and s_4 with a negative real part, there is an oscillation although it is convergent (damped).

2) Roots s_k with a positive real part: $\mathrm{Re}[s_k] > 0$. The exponential functions tend to infinity with time. In general, there is not any oscillation. In case of oscillation, it will be divergent.

3) Two roots with a positive real part: $\mathrm{Re}[s_1, s_2] > 0$, and two roots with a negative real part: $\mathrm{Re}[s_3, s_4] < 0$. In this case, $x_p(t)$ and $x_d(t)$ increase with the time and, in general, there is not any oscillation. However, if s_1 and s_2 are two complex conjugate roots: $s_1, s_2 = \sigma \pm j\omega$, where σ is positive, then, neglecting the transient relative to the two roots s_3 and s_4 with a negative real part, there is an oscillation although it is divergent (amplified).

4) Two imaginary roots: $s_1, s_2 = \pm j\omega$ and two roots with a negative real part: $\mathrm{Re}[s_3, s_4] < 0$. In this case, if we neglect the transient response relative to the two roots with a negative real part, we have steady oscillations.

Therefore, so that the machine works at a cyclic steady state, it is necessary that the characteristic polynomial $W(s)$ has two imaginary roots and two roots with a negative real part. In particular, the absolute value of this real part must be very high to ensure a quick drop of the transient and, therefore, a quick arrival of the stationary oscillations for the reciprocating parts.

To establish under what conditions a fourth-degree polynomial with real coefficients has two imaginary roots and two roots with a negative real part, a theorem developed by the authors for this purpose may be applied. It is illustrated in the following section, and it is called "P4 Theorem."

P4 Theorem

Statement

So that a fourth-degree polynomial with real coefficients has two imaginary roots and two roots with a negative real part,

$$G(s) = s^4 + \alpha s^3 + \beta s^2 + \gamma s + \delta \tag{41}$$

its coefficients must satisfy the following constraints:

$$\alpha, \beta, \gamma, \delta > 0 \tag{42}$$

$$\beta - \gamma/\alpha = \delta\alpha/\gamma \tag{43}$$

In particular, the two imaginary roots are

$$\pm j\omega$$

where

$$\omega^2 = \gamma/\alpha \tag{44}$$

whereas the two roots with a negative real part are

$$(-\alpha \pm \sqrt{\Delta})/2 \tag{45}$$

where

$$\Delta = \alpha^2 - 4\delta\alpha/\gamma \tag{46}$$

Proof

Let us consider a fourth-degree polynomial, $P(s)$, with real coefficients, and let us assume that it has the following

1) Two roots s_1 and s_2 having the form: $s_1 = +\sqrt{-a}$, and $s_2 = -\sqrt{-a}$, where a is a real number and may be either positive or negative. In the former case, s_1 and s_2 are imaginary roots: $\pm j\sqrt{a}$. In the latter case, s_1 and s_2 are real roots: $\pm\sqrt{-a}$.

2) Two roots s_3 and s_4 having any form.

Therefore, the $P(s)$ polynomial will have the following expression:

$$P(s) = (s^2 + a)(s - s_3)(s - s_4)$$
$$= s^4 - (s_3 + s_4)s^3 + (a + s_3s_4)s^2 - a(s_3 + s_4)s + as_3s_4$$

So that the $G(s)$ polynomial, given by Eq. (41), has the same roots of the $P(s)$ polynomial, it is necessary that

$$\alpha = -(s_3 + s_4), \qquad \beta = a + s_3s_4$$
$$\gamma = -a(s_3 + s_4), \qquad \delta = as_3s_4$$

from which we obtain

$$\begin{bmatrix} s_3 + s_4 = -\alpha \\ s_3 + s_4 = -\gamma/a \end{bmatrix} \Rightarrow a = \gamma/\alpha \tag{47}$$

$$\begin{bmatrix} s_3s_4 = \beta - a \\ s_3s_4 = \delta/a \end{bmatrix} \Rightarrow \beta - a = \delta/a \tag{48}$$

Substituting Eq. (47) in Eq. (48), we get Eq. (43).

Therefore, if the α, β, γ, and δ coefficients of the $G(s)$ polynomial satisfy Eq. (43), the $G(s)$ polynomial has the same roots of the $P(s)$ polynomial. In particular, if α and γ have the same sign, then the real number a is positive [see Eq. (47)] and, therefore, s_1 and s_2 are imaginary roots: $\pm j\sqrt{a}$. If α and γ have the opposite sign then the real number a is negative [see Eq. (47)] and, therefore, s_1 and s_2 are real roots: $\pm\sqrt{-a}$.

To have insights into the remaining roots, s_3 and s_4, we may use the following relations:

$$s_3 + s_4 = -\alpha$$
$$s_3s_4 = \beta - \gamma/\alpha = \delta\alpha/\gamma$$

Thus, s_3 and s_4 are the roots of a second-degree polynomial having the following form:

$$g(s) = s^2 + \alpha s + \delta\alpha/\gamma \tag{49}$$

Applying the well-known quadratic formula, the roots s_3 and s_4 are given by Eq. (45), where the discriminant Δ is given by Eq. (46). If α and γ have the same sign, we can consider three cases, according to whether δ is greater than, less than, or equal to zero.

1) $\delta < 0 \Rightarrow \Delta > 0 \Rightarrow s_3$ and s_4 are real and different roots. In this case, one of the two roots is always positive, both if $\alpha > 0$ (and, therefore, $\gamma > 0$) and if $\alpha < 0$ (and, therefore, $\gamma < 0$).

2) $\delta > 0 \Rightarrow$ the discriminant Δ may be greater than, less than, or equal to zero. Therefore, the s_s and s_4 roots may be real and different roots ($\Delta > 0$), complex conjugate roots ($\Delta < 0$), or real and equal roots ($\Delta = 0$). In this case, if $\alpha > 0$, the s_3 and s_4 roots have a negative real part. Instead, if $\alpha < 0$, the s_3 and s_4 roots have a positive real part.

3) $\delta = 0 \Rightarrow \Delta = \alpha^2 \Rightarrow s_3$ and s_4 are real and different roots. In this case, if $\alpha > 0$, we have $s_3 = 0$ and $s_4 = -\alpha < 0$. Instead, if $\alpha < 0$, we have $s_3 = 0$ and $s_4 = -\alpha > 0$.

So that the roots s_3 and s_4 have a negative real part, it is necessary that δ and α are positive. If α is positive, then also γ must be positive so that s_1 and s_2 are two imaginary roots. In this case, setting $a = \omega^2$, the two imaginary roots may be written as $\pm j\omega$. If α, γ, and δ must be positive, it follows from Eq. (43) that the coefficient β must also be positive.

Basic Criterion for Cyclic Steady-State: Laws of Motion of Piston and Displacer

The P4 Theorem applied to the characteristic polynomial $W(s)$ associated with the FPSE/LD–L system yields the conditions for the periodic steady operation of the machine. The $W(s)$ polynomial given by Eq. (25), with positions [Eqs. (26–29)], has to be written in the form of Eq. (41). Therefore, we have

$$\alpha = D'_{dd} + D'_{pp} \tag{50}$$

$$\beta = S'_{dd} + D'_{pp}D'_{dd} + S'_{pp} - D'_{dp}D'_{pd} \tag{51}$$

$$\gamma = D'_{pp}S'_{dd} + D'_{dd}S'_{pp} - D'_{dp}S'_{pd} - D'_{pd}S'_{dp} \tag{52}$$

$$\delta = S'_{pp}S'_{dd} - S'_{dp}S'_{pd} \tag{53}$$

Equations (42) and (43), whose coefficients α, β, γ, and δ depend on the S'_{ij} and D'_{ij} coefficients as listed earlier, define the basic criterion for the cyclic steady state of the machine.

Note that this criterion does not give any information about the stable operation of a free-piston Stirling machine. In fact, the stable operation may be considered as the capability of the machine to adjust its response to changes in the operational parameters without requiring external power controls, whereas the oscillation criterion, expressed by Eqs. (42) and (43), ensures (if verified) only the steady oscillations of the moving elements, once the operating conditions have been fixed. The stability analysis of the machine is developed in a companion paper.[9]

If a free-piston machine is running at a periodic steady state, which depends on the adopted operational parameters, then Eqs. (42) and (43) are satisfied. Therefore, the laws of motion of the piston and displacer are given by either Eqs. (30) and (31) or Eqs. (39) and (40), by simply setting s_1, $s_2 = \pm j\omega$, and s_3, $s_4 = (-\alpha \pm \sqrt{\Delta})/2$.

The calculation of the constants of integration c_k ($k = 1, \ldots, 4$), appearing in Eqs. (30) and (31), is much more difficult than the calculation of the residues R_{pk} and R_{dk} ($k = 1, \ldots, 4$), appearing in Eqs. (39) and (40). In fact, it is a lengthy matter to solve analytically four nonhomogeneous linear simultaneous algebraic equations [obtained by imposing the initial conditions for the piston and displacer in Eqs. (30 and (31)] in four unknowns, c_1, c_2, c_3, and c_4. This mathematical aspect shows the advantage of the Laplace transform with respect to the integration in the time domain for solving the FPSE/LD–L dynamic equations.

Therefore, by using Eqs. (39) and (40), where the transient response relative to the s_3 and s_4 roots having a negative real part may be neglected, the laws of motion of the moving elements are

$$x_p(t) = R_{p1}e^{j\omega t} + R_{p2}e^{-j\omega t} \tag{54}$$

$$x_d(t) = R_{d1}e^{j\omega t} + R_{d2}e^{-j\omega t} \tag{55}$$

Bearing in mind $e^{\pm j\omega t} = \cos\omega t \pm j \sin \omega t$, and applying the compound angle formula for sine of the difference of two angles, the preceding laws of motion may be written as

$$x_p(t) = (X_p/2)\sin(\omega t - \phi) \tag{56}$$

$$x_d(t) = (rX_p/2)\sin(\omega t) \tag{57}$$

where

$$X_p = 4(R_{p1}R_{p2})^{1/2} \tag{58}$$

$$r = \left(\frac{R_{d1}R_{d2}}{R_{p1}R_{p2}}\right)^{1/2} \tag{59}$$

$$\phi = \tan^{-1}\left(j\frac{R_{p1} + R_{p2}}{R_{p1} - R_{p2}}\right) - \tan^{-1}\left(j\frac{R_{d1} + R_{d2}}{R_{d1} - R_{d2}}\right) \tag{60}$$

The operating angular frequency ω of the machine is given by Eq. (44). From Eqs. (56) and (57) it follows that the dynamic quantities X_p, r, ϕ, and ω, univocally fix the laws of motion of the piston and displacer.

The calculation of the residues R_{p1}, R_{p2}, R_{d1}, and R_{d2} may be obtained applying Eq. (A2), where $G(s)/P(s)$ must be replaced by Eq. (35) for the piston residues, and by Eq. (36) for the displacer residues. Because the laws of motion in the Laplace domain $\tilde{x}_p(s)$ and $\tilde{x}_d(s)$ depend on $\zeta_{p0}(s)$ and $\zeta_{d0}(s)$, given by Eqs. (33) and (34), respectively, the calculation requires the knowledge of the initial conditions for the piston and displacer.

When the machine is not working, the piston and displacer velocities as well as their displacements are equal to zero. Therefore, if the machine starts at $t = 0$, the initial conditions are

$$x_{p0} = x_{d0} = \dot{x}_{p0} = \dot{x}_{d0} = 0 \tag{61}$$

The starting of the machine requires that one or more operational parameters are time dependent. But in the present study we consider only the operation of FPSE/LD–L systems with constant operating conditions. It follows that the initial conditions will never be given by Eq. (61). Therefore, the initial conditions represent the displacements and velocities of the moving elements at any time during engine operating, provided the operating conditions are time independent. In other words, any time can be assumed as an initial time ($t = 0$). In addition, if Eqs. (42) and (43) are satisfied, then the laws of motion [Eqs. (56) and (57)] are valid, and from these laws it follows that

$$x_{p0} = -(X_p/2)\sin\phi, \qquad x_{d0} = 0 \tag{62}$$

$$\dot{x}_{p0} = (X_p/2)\omega\cos\phi, \qquad \dot{x}_{d0} = (X_p/2)r\omega \tag{63}$$

Once the initial conditions for the moving elements have been established, the piston and displacer residues may be evaluated, and the results are shown in Appendix B.

Substituting the expressions obtained for the piston and displacer residues, given by Eqs. (B1) and (B2), in Eqs. (58), (59), and (60), we have, respectively,

$$1 = 2\left(\frac{A^2 + B^2}{p^2 + q^2}\right)^{1/2} \tag{64}$$

$$r = \left(\frac{C^2 + D^2}{A^2 + B^2}\right)^{1/2} \tag{65}$$

$$\phi = \tan^{-1}\left(\frac{Ap + Bq}{Bp - Aq}\right) - \tan^{-1}\left(\frac{Cp + Dq}{Dp - Cq}\right) \tag{66}$$

where the coefficients A, B, C, D, p, and q, depend on r, ϕ, and ω, and on the S'_{ij} and D'_{ij} coefficients per unit of moving element mass of the FPSE/LD–L system, as shown in Appendix B. Because Eq. (66) is quite complex, with suitable manipulations it is possible to write it in the following simpler form:

$$\tan\phi = \frac{AD - BC}{AC + BD} \tag{67}$$

Equations (42–44), (64), (65), (67) represent the dynamic behavior of any FPSE/LD–L system. Instead, the influence of a particular LD–L subsystem as well as the thermodynamic behavior and the heat-exchange processes of a specific FPSE are described by means of the S'_{ij} and D'_{ij} coefficients. These equations may be considered as the basic equations for the study of FPSE/LD–L systems.

Bearing in mind Eqs. (16), (56), and (57), the casing motions will be sinusoidal

$$x_c(t) = (r_c X_p/2)\sin(\omega t - \phi_c)$$

where

$$r_c = \sqrt{\mu_{pc}{}^2 + 2\mu_{pc}\mu_{dc} r\cos\phi + (r\mu_{dc})^2}$$

$$\tan\phi_c = \frac{\mu_{pc}\sin\phi}{\mu_{pc}\cos\phi + \mu_{dc} r}$$

Conclusions

The analysis reported in the previous sections shows how the FPSE dynamic balance equations can be solved, by means of a suitable linearization technique, both in the time domain and in the Laplace domain.

Because the solution can be expressed in an analytical form as a function of the stiffness and damping coefficients of the machine, it is possible to derive the conditions that the listed coefficients have to satisfy to ensure an engine's cyclic steady operation.

These conditions may be expressed in a mathematically rigorous form as the consequence of a theorem developed by the authors for this subject and called P4 Theorem.

In addition, it has been proven that a self-centering of the moving elements without centering devices may not be obtained if the engine uses a clearance seal/flexure bearing system for the piston and displacer. Therefore, in the authors' opinion, a centering device for the moving elements should always be foreseen in a free-piston machine.

Appendix A: Partial Fractions

To resolve an algebraic expression having the form $G(s)/P(s)$ into partial fractions: 1) the denominator $P(s)$ must factorize and 2) the numerator $G(s)$ must be at least one degree less than the denominator $P(s)$. When the degree of the numerator is equal to or higher than the degree of the denominator, the numerator must be divided by the denominator until the remainder is of less degree than the denominator.

There are basically two types of partial fractions and the form of partial fraction used is described briefly as follows, where $G(s)$ is assumed to be of less degree than $P(s)$.

If all of the roots of the denominator polynomial have multiplicity $m = 1$, then we have

$$\frac{G(s)}{P(s)} = \sum_{k=1}^{n} \frac{R_k}{s - s_k} \tag{A1}$$

where n is the degree of $P(s)$ polynomial, s_k is the kth root of $P(s)$, and R_k is the residue associated with s_k, given by

$$R_k = \lim_{s\to s_k}(s - s_k)\frac{G(s)}{P(s)} \tag{A2}$$

On the contrary, if some root s_k of $P(s)$ polynomial has multiplicity $m_k > 1$, then Eq. (A1) has to be replaced by

$$\frac{G(s)}{P(s)} = \sum_{k=1}^{r}\left[\sum_{\alpha=1}^{m_k}\frac{R_{k\alpha}}{(s - s_k)^{\alpha}}\right] \tag{A3}$$

where r is the number of the roots of $P(s)$ polynomial and $R_{k\alpha}$ is the residue associated with s_k, given by

$$R_{k\alpha} = \lim_{s\to s_k}\left\{\left(\frac{1}{m_k - \alpha!}\right)\frac{d^{m_k-\alpha}}{ds^{m_k-\alpha}}\left[(s - s_k)\frac{G(s)}{P(s)}\right]\right\} \tag{A4}$$

where $\alpha = 1, \ldots, m_k$.

Appendix B: Piston and Displacer Residues

The piston R_{p1}, R_{p2} and displacer R_{d1}, R_{d2} residues, appearing in Eqs. (58–60), may be calculated applying Eq. (A2), where $G(s)/P(s)$ has to be replaced by Eq. (35) for the piston residues, and by Eq. (36) for the displacer residues. Therefore, bearing in mind the initial conditions [Eqs. (62) and (63)] for the moving elements, we have

$$R_{p1}, R_{p2} = \frac{A \pm jB}{p \pm jq}\left(\frac{X_p}{2}\right) \tag{B1}$$

$$R_{d1}, R_{d2} = \frac{C \pm jD}{p \pm jq}\left(\frac{X_p}{2}\right) \tag{B2}$$

where the + sign is valid for R_{p1} and R_{d1}, whereas the − sign is valid for R_{p2} and R_{d2}. The coefficients appearing in the preceding expressions are

$$\begin{aligned}
A &= (D'_{dd}S'_{pp} - D'_{pd}S'_{dp})\sin\phi + \omega(S'_{dd} - \omega^2)\cos\phi - r\omega S'_{pd} \\
B &= \omega[(S'_{pp} - \delta/\omega^2)\sin\phi + \omega D'_{dd}\cos\phi - r\omega D'_{pd}] \\
C &= (D'_{pp}S'_{dp} - D'_{dp}S'_{pp})\sin\phi + r\omega(S'_{pp} - \omega^2) - \omega S'_{dp}\cos\phi \\
D &= \omega[S'_{dp}\sin\phi - D'_{dp}\omega\cos\phi + r\omega D'_{pp}] \\
p &= -2\alpha\omega^2 \\
q &= 2\omega(\delta/\omega^2 - \omega^2)
\end{aligned}$$

where α and δ are given, respectively, by Eqs. (50) and (53).

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